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International Baccalaureate®
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MATHEMATICS
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PAPER 3 – SETS, RELATIONS AND GROUPS

Thursday 20 May 2010 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 10]*

The function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x) = 2e^{x} - e^{-x}.$$

(a) Show that f is a bijection. *[4 marks]*

(b) Find an expression for $f^{-1}(x)$. *[6 marks]*

2. *[Maximum mark: 10]*

The relation R is defined for 2×2 matrices such that $\boldsymbol{A}R\boldsymbol{B}$ if and only if there exists a non-singular matrix $\boldsymbol{H}$ such that $\boldsymbol{AH} = \boldsymbol{HB}$.

(a) Show that R is an equivalence relation. *[7 marks]*

(b) Given that $\boldsymbol{A}$ is singular and $\boldsymbol{A}R\boldsymbol{B}$, show that $\boldsymbol{B}$ is also singular. *[3 marks]*

3. *[Maximum mark: 14]*

(a) Consider the set $A = \{1, 3, 5, 7\}$ under the binary operation $*$, where $*$ denotes multiplication modulo 8.

(i) Write down the Cayley table for $\{A, *\}$.

(ii) Show that $\{A, *\}$ is a group.

(iii) Find all solutions to the equation $3 * x * 7 = y$. Give your answers in the form (x, y). *[9 marks]*

(This question continues on the following page)

(Question 3 continued)

(b) Now consider the set $B = \{1, 3, 5, 7, 9\}$ under the binary operation $\otimes$, where $\otimes$ denotes multiplication modulo 10. Show that $\{B, \otimes\}$ is not a group. *[2 marks]*

(c) Another set C can be formed by removing an element from B so that $\{C, \otimes\}$ is a group.

(i) State which element has to be removed.

(ii) Determine whether or not $\{A, *\}$ and $\{C, \otimes\}$ are isomorphic. *[3 marks]*

4. *[Maximum mark: 13]*

The permutation p_1 of the set $\{1, 2, 3, 4\}$ is defined by

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

(a) (i) State the inverse of p_1.

(ii) Find the order of p_1. *[5 marks]*

(b) Another permutation p_2 is defined by

$$p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

(i) Determine whether or not the composition of p_1 and p_2 is commutative.

(ii) Find the permutation p_3 which satisfies

$$p_1 p_3 p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

[8 marks]

5. *[Maximum mark: 13]*

Let G be a finite cyclic group.

(a) Prove that G is Abelian. *[4 marks]*

(b) Given that a is a generator of G, show that a^{-1} is also a generator. *[5 marks]*

(c) Show that if the order of G is five, then all elements of G, apart from the identity, are generators of G. *[4 marks]*